Turaev-Viro-Barrett-Westbury Invariants

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von **Tessa Kammermeier**

Betreuerin: Prof. Dr. Catherine Meusburger

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE
FAKULTÄT

Department
MATHEMATIK

Introduction

This thesis presents an algebraic framework for constructing invariants of closed oriented 3-manifolds by taking a state sum model on a triangulation. This method has first been described by John W. Barrett and Bruce W. Westbury and was based on work by Vladimir G. Turaev and Oleg Y. Viro. The underlying algebraic framework consists of a spherical fusion category, that is, a $\mathrm{Vect}_{\mathbb{k}}^{fd}$-enriched abelian monoidal category with a notion of semisimplicity in which we can take a generalised trace of an endomorphism. From this category, we can construct the 6j-symbols, which are determined by the associator of the monoidal category. By assigning these 6-symbols to the triangulation of a manifold and summing over all possible assignments, we get the Turaev-Viro-Barrett-Westbury invariant of the manifold. It only depends on the homeomorphism class and orientation of the underlying manifold. Lastly, we will calculate a Turaev-Viro-Barrett-Westbury invariant of two manifolds explicitly and by that show that they can distinguish topological spaces with the same homology and homotopy groups.

Historically, this type of construction was first done by Turaev and Viro in [TV92], where they used a spherical fusion category constrcuted from $U_q(\mathfrak{sl}_2(\mathbb{C}))$ as the category from which they constructed the 6j-symbols. Later, in [BW96] Barrett and Westbury generalised this construction to spherical fusion categories which they defined in [BW99]. Turaev himself also generalised his construction in [Tur16] to work with so called unimodal categories. The calculations for lens spaces $L(p,q)$ and $\mathrm{Vect}_G^{\omega,fd}$ which we will perform at the end of this thesis have in part already be done in [AC93], as the Turaev-Viro-Barrett-Westbury invariants for the category $\mathrm{Vect}_G^{\omega,fd}$ resemble the Dijkgraaf-Witten partition functions of a closed 3-manifold as defined in [DW90].

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1 Spherical Fusion Categories

In the following, we define the necessary categorical framework for this thesis. Most of these definitions are taken from [EGNO15] chapters 1 and 4. First, we define a type of semisimplicity for categories. We assume $\mathbb{k}$ to be an algebraically closed field.

Definition 1.1. An additive category $\mathcal{C}$ is called $\mathbb{k}$-*linear* if, for all objects x,y in $\mathcal{C}$, the set $\mathrm{Hom}(x,y)$ is equipped with a vector space structure over $\mathbb{k}$ such that composition of morphisms is $\mathbb{k}$-linear.

Definition 1.2. Let $\mathcal{C}$ be an abelian category. A *subobject* of an object x is a monomorphism with codomain x. A nonzero object x in $\mathcal{C}$ is called *simple* if $0: 0 \to x$ and $\mathrm{id}_x : x \to x$ are its only subobjects up to isomorphism in the slice category $\mathcal{C}/x$. An object x in $\mathcal{C}$ is called *semisimple* if it can be written as a direct sum of finitely many simple objects. $\mathcal{C}$ is called *semisimple* if every object in $\mathcal{C}$ is semisimple.

Definition 1.3. A $\mathbb{k}$-linear abelian category $\mathcal{C}$ is *finite* if for any two objects x,y in $\mathcal{C}$, the $\mathbb{k}$-vector space $\mathrm{Hom}(x,y)$ is finite dimensional and there are only finitely many isomorphism classes of simple objects.

[EGNO15] requires more of category to be considered finite but we won't need the additional requirements in this thesis. These finite categories behave in some ways similarly to the category of representations of a group. We can even generalise the well known Schur's lemma for these categories. We fix a set I of representatives of simple objects.

Lemma 1.4 (Schur's lemma)**.** *Let $\mathcal{C}$ be a finite category and a,b simple objects in $\mathcal{C}$. We have $\mathrm{Hom}(a,b) = 0$ if a and b are non-isomorphic and $\mathrm{Hom}(a,a) = \mathbb{k}$.*

Proof. Let $f : a \to b$ be a nonzero morphism. Because $\iota : \ker(f) \to a$ is a monomorphism and a is simple, $\ker(f)$ has to be 0 and f is a monomorphism. As b is simple and a is nonzero by definition, f is isomorphic to id_b in $\mathcal{C}/b$ and therefore an isomorphism. This implies that $\mathrm{Hom}(a,a)$ is a finite-dimensional division algebra over $\mathbb{k}$, which is algebraically closed, and therefore $\mathrm{Hom}(a,a) = \mathbb{k}$. □

The simple objects have the property that each morphism can be uniquely factorised through them in the following sense.

Lemma 1.5. *Let $\mathcal{C}$ be a finite semisimple category and x,y objects in $\mathcal{C}$. Then the morphism*

$$\bigoplus_{a \in I} \mathrm{Hom}(x,a) \otimes \mathrm{Hom}(a,y) \to \mathrm{Hom}(x,y)$$
$$\sum_{a \in I} f_a \otimes g_a \mapsto \sum_{a \in I} g_a f_a$$

is an isomorphism of vector spaces.

Proof. As $\mathcal{C}$ is semisimple, we have $x \cong \bigoplus_{a\in I} a^{\oplus n_a}$ and we can write id_x as the sum $\sum_{a\in I}\sum_{k=1}^{n_a} \iota_a^k \pi_a^k$, where $\iota_a^k : a \to x$ and $\pi_a^k : x \to a$ are the injections and projections into and respectively from the k-th component of a in x. We can now write every morphism $f : x \to y$ as $f\,\mathrm{id}_x$ and therefore $\sum_{a\in I}\sum_{k=1}^{n_a} \pi_a^k \otimes (f\iota_a^k)$ maps onto f which implies surjectivity. The injectivity follows by counting dimensions. As $\mathcal{C}$ is semisimple, we can write x as $\bigoplus_{b\in I} b^{\oplus n_b^x}$ and y as $\bigoplus_{b\in I} b^{\oplus n_b^y}$. By plugging these into the equation above and taking the dimension we get

$$
\begin{aligned}
&\dim\left(\bigoplus_{a\in I}\mathrm{Hom}\left(\bigoplus_{b\in I} b^{\oplus n_b^x}, a\right)\otimes \mathrm{Hom}\left(a, \bigoplus_{b\in I} b^{\oplus n_b^y}\right)\right)\\
&= \sum_{a\in I}\dim\left(\mathrm{Hom}\left(a^{\oplus n_a^x}, a\right)\otimes \mathrm{Hom}\left(a, a^{\oplus n_a^y}\right)\right)\\
&= \sum_{a\in I}\dim\left(\mathrm{Hom}(a,a)^{\oplus n_a^x}\otimes \mathrm{Hom}(a,a)^{\oplus n_a^y}\right) = \sum_{a\in I}\dim\left(\Bbbk^{\oplus n_a^x}\otimes \Bbbk^{\oplus n_a^y}\right) = \sum_{a\in I} n_a^x n_a^y,\\
&\dim\mathrm{Hom}\left(\bigoplus_{b\in I} b^{\oplus n_b^x}, \bigoplus_{b\in I} b^{\oplus n_b^y}\right) = \sum_{a\in I}\sum_{b\in I}\dim\mathrm{Hom}\left(a^{\oplus n_a^x}, b^{\oplus n_b^y}\right)\\
&= \sum_{a\in I}\dim\mathrm{Hom}\left(a^{\oplus n_a^x}, a^{\oplus n_a^y}\right) = \sum_{a\in I} n_a^x n_a^y,
\end{aligned}
$$

where we used Schur's lemma and the fact that direct sums commute with the Hom-functor. □

We now introduce the necessary requirements for the monoidal structure on our category and show how these interact with the framework we defined above.

Definition 1.6. Let $\mathcal{C}$ be a monoidal category.

An object x in $\mathcal{C}$ is called *right dualisable* if there is an object x^* and morphisms

$$\mathrm{ev}_x^R : x^*\otimes x \to \mathbb{1} \qquad \mathrm{coev}_x^R : \mathbb{1} \to x\otimes x^*$$

such that the following diagrams commute:

$$
\begin{array}{ccccc}
x & \xrightarrow{l_x^{-1}} & \mathbb{1}\otimes x & \xrightarrow{\mathrm{coev}_x^R\otimes \mathrm{id}_x} & (x\otimes x^*)\otimes x\\
\downarrow{\scriptstyle \mathrm{id}_x} & & & & \downarrow{\scriptstyle a_{x,x^*,x}}\\
x & \xleftarrow[r_x]{} & x\otimes \mathbb{1} & \xleftarrow[\mathrm{id}_x\otimes \mathrm{ev}_x^R]{} & x\otimes(x^*\otimes x)
\end{array}
\qquad
\begin{array}{ccccc}
x^* & \xrightarrow{r_{x^*}^{-1}} & x^*\otimes \mathbb{1} & \xrightarrow{\mathrm{id}_{x^*}\otimes \mathrm{coev}_x^R} & x^*\otimes(x\otimes x^*)\\
\downarrow{\scriptstyle \mathrm{id}_{x^*}} & & & & \downarrow{\scriptstyle a^{-1}_{x^*,x,x^*}}\\
x^* & \xleftarrow[l_{x^*}]{} & \mathbb{1}\otimes x^* & \xleftarrow[\mathrm{ev}_x^R\otimes \mathrm{id}_{x^*}]{} & (x^*\otimes x)\otimes x^*
\end{array}
$$

An object x in $\mathcal{C}$ is called *left dualisable* if there is an object ${}^*x$ and morphisms

$$\mathrm{ev}_x^L : x\otimes {}^*x \to \mathbb{1} \qquad \mathrm{coev}_x^L : \mathbb{1} \to {}^*x\otimes x$$

such that the following diagrams commute:

$$\begin{array}{ccccc} x & \xrightarrow{r_x^{-1}} & x\otimes\mathbb{1} & \xrightarrow{\mathrm{id}_x\otimes\mathrm{coev}_x^L} & x\otimes({}^*x\otimes x) \\ \downarrow{\scriptstyle \mathrm{id}_x} & & & & \downarrow{\scriptstyle a^{-1}_{x,{}^*x,x}} \\ x & \xleftarrow[l_x]{} & \mathbb{1}\otimes x & \xleftarrow[\mathrm{ev}_x^L\otimes\mathrm{id}_x]{} & (x\otimes{}^*x)\otimes x \end{array} \qquad \begin{array}{ccccc} {}^*x & \xrightarrow{l^{-1}_{{}^*x}} & \mathbb{1}\otimes{}^*x & \xrightarrow{\mathrm{coev}_x^L\otimes\mathrm{id}_{{}^*x}} & ({}^*x\otimes x)\otimes{}^*x \\ \downarrow{\scriptstyle \mathrm{id}_{{}^*x}} & & & & \downarrow{\scriptstyle a_{{}^*x,x,{}^*x}} \\ {}^*x & \xleftarrow[r_{{}^*x}]{} & {}^*x\otimes\mathbb{1} & \xleftarrow[\mathrm{id}_{{}^*x}\otimes\mathrm{ev}_x^L]{} & {}^*x\otimes(x\otimes{}^*x) \end{array}$$

The category $\mathcal{C}$ is called *right rigid* if every object in $\mathcal{C}$ is right dualisable, *left rigid* if every object is left dualisable and *rigid* if it is both right and left rigid.

Remark 1.7. Right duals in a right rigid monoidal category $\mathcal{C}$ define a monoidal functor $*:\mathcal{C}\to\mathcal{C}^{op,op}$, where $\mathcal{C}^{op,op}$ is the monoidal category with opposite composition and opposite tensor product. $*$ assigns the right dual x^* to an object x and maps $f:x\to y$ onto

$$f^*: y^* \xrightarrow{r^{-1}_{y^*}} y^*\otimes\mathbb{1} \xrightarrow{\mathrm{id}_{y^*}\otimes\mathrm{coev}_x^R} y^*\otimes(x\otimes x^*) \xrightarrow{\mathrm{id}_{y^*}\otimes(f\otimes\mathrm{id}_{x^*})} y^*\otimes(y\otimes x^*) \xrightarrow{a^{-1}_{y^*,y,x^*}} (y^*\otimes y)\otimes x^*$$
$$\xrightarrow{\mathrm{ev}_y^R\otimes\mathrm{id}_{x^*}} \mathbb{1}\otimes x^* \xrightarrow{l_{x^*}} x^*.$$

Definition 1.8. Let $\mathcal{C}$ be a finite semisimple rigid monoidal category. $\mathcal{C}$ is a *fusion category* if the functor $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is bilinear on morphisms and $\mathrm{End}(\mathbb{1})\cong\Bbbk$.

Lemma 1.9. *Let $\mathcal{C}$ be a fusion category. Then, the object $\mathbb{1}$ is simple.*

Proof. If we plug $x=y=\mathbb{1}$ into the isomorphism from Lemma 1.5 we get

$$\bigoplus_{a\in I}\mathrm{Hom}(\mathbb{1},a)\otimes\mathrm{Hom}(a,\mathbb{1})\cong\mathrm{End}(\mathbb{1})\cong\Bbbk.$$

By counting dimensions we see there has to be exactly one $a\in I$ such that $\dim\mathrm{Hom}(\mathbb{1},a)=1$ and $\dim\mathrm{Hom}(\mathbb{1},b)=0$ for all $b\in I$ with $b\neq a$. As $\mathbb{1}$ can be written as a direct sum of simple objects, this implies that $\mathbb{1}\cong a$. □

This also proves that, that any object x with $\mathrm{Hom}(x,x)\cong\Bbbk$ is simple.

Definition 1.10. Let $\mathcal{C}$ be a right monoidal category. A *pivotal structure* on $\mathcal{C}$ is a monoidal natural isomorphism $\omega:\mathrm{id}_{\mathcal{C}}\to **$. A *pivotal category* is a pair $(\mathcal{C},\omega)$ of a right rigid monoidal category category $\mathcal{C}$ and a pivotal structure ω on $\mathcal{C}$.

Remark 1.11. Every pivotal category is left rigid, and right dual objects in a pivotal category are left dual objects.

We define ${}^*x:=x^*$ for all objects x of $\mathcal{C}$ and

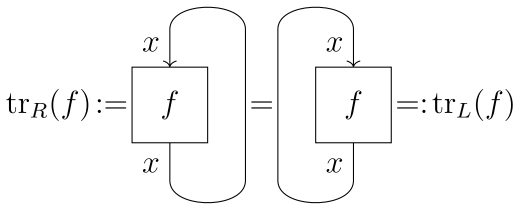
$$\mathrm{ev}_x^L: x\otimes x^* \xrightarrow{\omega\otimes\mathrm{id}_{x^*}} x^{**}\otimes x^* \xrightarrow{\mathrm{ev}^R_{x^*}} \mathbb{1} \qquad \mathrm{coev}_x^L:\mathbb{1} \xrightarrow{\mathrm{coev}^R_{x^*}} x^*\otimes x^{**} \xrightarrow{\mathrm{id}_{x^*}\otimes\omega^{-1}} x^*\otimes x.$$

One can show that this implies that every object is left dualisable.

For pivotal categories $\mathcal{C}$ we have the following *graphical calculus*. Objects in $\mathcal{C}$ are represented by vertical lines with an orientation, where a downwards pointing line labelled with x represents the object x and an upwards pointing line labelled with x represents the dual x^*. A morphism $f: x \to y$ is represented as a vertex on a vertical line that divides the line into an upper part labelled x and a lower part labelled y. Identity morphisms, the tensor unit $\mathbb{1}$, unit morphisms $f: \mathbb{1} \to \mathbb{1}$ and the associator are not represented in these diagrams. The composition of morphisms is given by the vertical composition of diagrams and the tensor products of objects and morphisms are given by horizontal composition of diagrams. The evaluations and coevaluations are given by

$\mathrm{ev}_x^R: x^* \otimes x \to \mathbb{1}$ $\qquad$ $\mathrm{coev}_x^R: \mathbb{1} \to x \otimes x^*$ $\qquad$ $\mathrm{ev}_x^L: x \otimes x^* \to \mathbb{1}$ $\qquad$ $\mathrm{coev}_x^L: \mathbb{1} \to x^* \otimes x$

Definition 1.12. A pivotal category $\mathcal{C}$ is called *spherical* if, for all objects x in $\mathcal{C}$ and all $f \in \mathrm{End}(x)$, the right and the left trace of f are equal:



We denote $\mathrm{tr}(f) := \mathrm{tr}_R(f) = \mathrm{tr}_L(f)$ and $\dim_q(x) := \mathrm{tr}(\mathrm{id}_x)$.

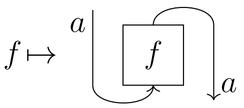
Definition 1.13. Let $\mathcal{C}$ be a spherical fusion category. The *dimension of* $\mathcal{C}$ is given by

$$\dim(\mathcal{C}) := \sum_{a \in I} \dim_q(a)^2.$$

Lemma 1.14. *Let $\mathcal{C}$ be a spherical fusion category. For any simple object a, the dual a^* is also simple and* $\dim_q(a) = \dim_q(a^*)$.

Proof. Via the evaluation and coevaluation, we get a canonical isomorphism

$$\mathrm{Hom}(a^*, a^*) \to \mathrm{Hom}(a, a)$$



Therefore $\mathrm{Hom}(a^*, a^*) \cong \mathrm{Hom}(a, a) \cong \Bbbk$. Furthermore $\dim_q(a^*) = \mathrm{tr}(\mathrm{id}_{a^*}) = \mathrm{tr}(\mathrm{id}_a^*) = \mathrm{tr}(\mathrm{id}_a) = \dim_q(a)$. □

Lemma 1.15. *Let $\mathcal{C}$ be a spherical fusion category. For any two objects x, y in $\mathcal{C}$, there is a non-degenerate bilinear pairing*

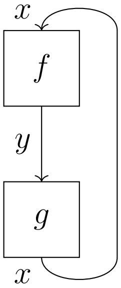
$$\Theta: \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, x) \to \Bbbk$$

defined by $\Theta(f, g) = \mathrm{tr}(fg) = \mathrm{tr}(gf)$.

Proof. First we note that it is sufficient to prove this for $x=\mathbb{1}$ as we have canonical isomorphisms

$$\begin{aligned}\alpha:&\mathrm{Hom}(x,y)\to\mathrm{Hom}(\mathbb{1},y\otimes x^*)\\ &\alpha(f)=(f\otimes\mathrm{id}_{x^*})\mathrm{coev}_x^R \quad \text{and}\\ \beta:&\mathrm{Hom}(y,x)\to\mathrm{Hom}(y\otimes x^*,\mathbb{1})\\ &\beta(g)=\mathrm{ev}_x^L(g\otimes\mathrm{id}_{x^*}).\end{aligned}$$

The trace is invariant under these morphisms: $\mathrm{tr}(\beta(g)\alpha(f))=\beta(g)\alpha(f)=\mathrm{tr}(gf)$. The first equality follows from the fact that $\mathrm{tr}(h)=h$ for all $h\in\mathrm{End}(\mathbb{1})$, the second equality from the definition of the trace:

$\beta(g)\alpha(f)=$  $=\mathrm{tr}(gf)$

Θ is non-degenerate if the adjoint morphisms $\theta_1:\mathrm{Hom}(\mathbb{1},y)\to\mathrm{Hom}(y,\mathbb{1})^*$ and $\theta_2:\mathrm{Hom}(y,\mathbb{1})\to\mathrm{Hom}(\mathbb{1},y)^*$ are isomorphisms. As $\mathcal{C}$ is semisimple, we can write id_y as $\sum_{a\in I}\sum_{k=1}^{n_a}\iota_a^k\pi_a^k$. For $f\in\mathrm{Hom}(\mathbb{1},y)$, we have

$$f=\sum_{a\in I}\sum_{k=1}^{n_a}\iota_a^k\pi_a^k f=\sum_{k=1}^{n_{\mathbb{1}}}\iota_{\mathbb{1}}^k\pi_{\mathbb{1}}^k f=\sum_{k=1}^{n_{\mathbb{1}}}\iota_{\mathbb{1}}^k\mathrm{tr}(\pi_{\mathbb{1}}^k f)$$

The second equality follows from Schur's lemma, the third from the fact that $\mathrm{tr}(h)=h$ for all $h\in\mathrm{End}(\mathbb{1})$, so if f is nonzero, one has that $\mathrm{tr}(\pi_{\mathbb{1}}^k f)$ is nonzero for some k. This implies the injectivity of θ_1. The surjectivity follows from counting dimensions. The proof for θ_2 is analogous. □

We will see in the next two lemmas that the tensor unit $\mathbb{1}$ acts as a sort of universal translator between the different but isomorphic endomorphism fields of simple objects if we regard the elements of $\mathrm{End}(\mathbb{1})$ as elements of the field $\mathbb{k}$. Intuitively, one can think of the trace translating an endomorphism $f\in\mathrm{End}(a)$ into $\mathrm{tr}(f)\in\mathrm{End}(\mathbb{1})$.

Lemma 1.16. *Let $\mathcal{C}$ be a spherical fusion category. For any simple object a in $\mathcal{C}$, $\dim_q(a)=\mathrm{tr}(\mathrm{id}_a)$ is nonzero.*

Proof. With respect to the generator id_a in $\mathrm{Hom}(a,a)\cong\mathbb{k}$, the paring $\Theta:\mathrm{Hom}(a,a)\times\mathrm{Hom}(a,a)\to\mathbb{k}$ from Lemma1.15 is represented by the 1×1-matrix $(\mathrm{tr}(\mathrm{id}_a))$. Non-degeneracy implies that $\mathrm{tr}(\mathrm{id}_a)=\dim_q(a)$ is nonzero. □

Lemma 1.17. *Let $\mathcal{C}$ be a spherical fusion category, a a simple object and x another arbitrary object. Pre- and post-composition with an endomorphism $f:a\to a$ acts on the vector spaces $\mathrm{Hom}(a,x)$ and $\mathrm{Hom}(x,a)$ respectively as multiplication by the scalar $\dim_q(a)^{-1}\mathrm{tr}(f)\in\mathrm{End}(\mathbb{1})\cong\mathbb{k}$.*

Proof. Because $\operatorname{End}(a)\cong\Bbbk$, we have $f=\lambda\mathrm{id}_a$ for a $\lambda\in\Bbbk$. Applying tr to both sides, we get $\lambda\dim_q(a)=\operatorname{tr}(f)$ and therefore $\lambda=\dim_q(a)^{-1}\operatorname{tr}(f)$. □

Finally, this last Lemma will become necessary much later in this thesis, when we prove that the Turaev-Viro-Barrett-Westbury invariants actually define invariants.

Lemma 1.18. *Let $\mathcal{C}$ be a spherical fusion category. For any object x in $\mathcal{C}$, we have*

$$\dim_q(x)=\sum_{a\in J}\dim_q(a)\dim\operatorname{Hom}(a,x).$$

Proof. As $\mathcal{C}$ is semisimple, we can write x as the direct sum $\bigoplus_{b\in I}b^{\oplus n_b}$. We note that one has

$$\operatorname{Hom}(a,x)=\operatorname{Hom}\left(a,\bigoplus_{b\in I}b^{\oplus n_b}\right)=\operatorname{Hom}\left(a,a^{\oplus n_a}\right)=\operatorname{Hom}(a,a)^{\oplus n_a}=\Bbbk^{n_a},$$

where the second and fourth equality follow from Schur's lemma. Therefore $n_a=\dim\operatorname{Hom}(a,x)$. We now write id_x as $\sum_{a\in I}\sum_{k=1}^{n_a}\iota_a^k\pi_a^k$ and apply the trace:

$$\begin{aligned}\dim_q(x)=\operatorname{tr}(\mathrm{id}_x)=\operatorname{tr}\left(\sum_{a\in I}\sum_{k=1}^{n_a}\iota_a^k\pi_a^k\right)&=\sum_{a\in I}\sum_{k=1}^{n_a}\operatorname{tr}(\iota_a^k\pi_a^k)=\sum_{a\in I}\sum_{k=1}^{n_a}\operatorname{tr}(\pi_a^k\iota_a^k)\\&=\sum_{a\in I}\sum_{k=1}^{n_a}\operatorname{tr}(\mathrm{id}_a)=\sum_{a\in I}\dim_q(a)n_a=\sum_{a\in I}\dim_q(a)\dim\operatorname{Hom}(a,x)\end{aligned}$$

□

1.1 The category $\mathbf{Vect}_G^{\omega,fd}$

We will now look at one example for a spherical fusion category.

Definition 1.19. Let G be a finite group and $\omega: G\times G\times G\to\mathbb{C}^\times$ a *3-cocycle*, a map that satisfies:

$$\omega(gh,k,l)\omega(g,h,kl)=\omega(g,h,k)\omega(g,hk,k)\omega(h,k,l) \quad \forall g,h,k,l\in G \tag{1}$$

The category $\mathrm{Vect}_G^{\omega,fd}$ of *finite-dimensional G-graded vector spaces over* $\mathbb{C}$ has

- finite-dimensional vector spaces over $\mathbb{C}$ with a decomposition $V=\bigoplus_{g\in G}V_g$ as objects and
- $\mathbb{C}$-linear maps $f: V\to W$ with $f(V_g)\subseteq W_g$ for all $g\in G$ as morphisms.

It is a monoidal category with the tensor product

$$V\otimes W=\bigoplus_{g\in G}(V\otimes W)_g \qquad (V\otimes W)_g=\bigoplus_{h\in G}V_h\otimes_{\mathbb{C}}W_{h^{-1}g}$$

and the associator given by the linear maps

$$\begin{aligned} a_{U_g,V_h,W_k}: (U_g\otimes_{\mathbb{C}}V_h)\otimes_{\mathbb{C}}W_k &\to U_g\otimes_{\mathbb{C}}(V_h\otimes_{\mathbb{C}}W_k) \\ (u\otimes v)\otimes w &\mapsto \omega(g,h,k)u\otimes(v\otimes w). \end{aligned}$$

One can easily see that the simple objects are $\mathbb{C}_g:=\bigoplus_{h\in G}\delta_{g,h}$, where $\delta_{g,h}=\mathbb{C}$ if $h=g$, and 0 otherwise. We will often just write g instead of $\mathbb{C}_g$ if it's clear what is meant. As G is finite, there are only a finite number of isomorphism classes of simple objects and $\mathrm{Vect}_G^{\omega,fd}$ is finite.

The category $\mathrm{Vect}_G^{\omega,fd}$ is right rigid, via

$$V^*=\left(\bigoplus_{g\in G}V_g\right)^*=\bigoplus_{g\in G}(V^*)_{g^{-1}},$$

and one can calculate that a pivotal structure $w_V: V\to V^{**}$ on $\mathrm{Vect}_G^{\omega,fd}$ is of the form

$$\begin{aligned} w_V: V_g &\to (V_g)^{**}=(V^{**})_g \\ v &\mapsto \beta(g)\Phi_v \end{aligned}$$

on the components of V, where $\Phi_v(f)=f(v)$ and $\beta: G\to\mathbb{C}^\times$ is a map such that $\kappa: G\to\mathbb{C}^\times$, $\kappa(g)=\beta(g)\omega(g,g^{-1},g)$ is a group homomorphism. The category $\mathrm{Vect}_G^{\omega,fd}$ is spherical iff $\kappa(g)=\kappa(g)^{-1}$ for all $g\in G$, that is if κ is a group homomorphism that takes values in $\{1,-1\}\subseteq\mathbb{C}$. This allows us to choose the trivial pivotal structure $\beta(g)=\omega(g,g^{-1},g)^{-1}$. With these choices, $\mathrm{Vect}_G^{\omega,fd}$ becomes a spherical fusion category with $\dim_q(\mathbb{C}_g)=1$ for all $g\in G$ and therefore $\dim(\mathrm{Vect}_G^{\omega,fd})=|G|$.

2 6j-Symbols

For the remainder of this thesis, $\mathcal{C}$ will be a spherical fusion category. In this chapter we will define the so called 6j-symbols and show the orthoganility and the Biedenharn-Elliot relation, which will be used to construct 3-manifold invariants. It mostly follows [Tur16] Chapter VI.1.

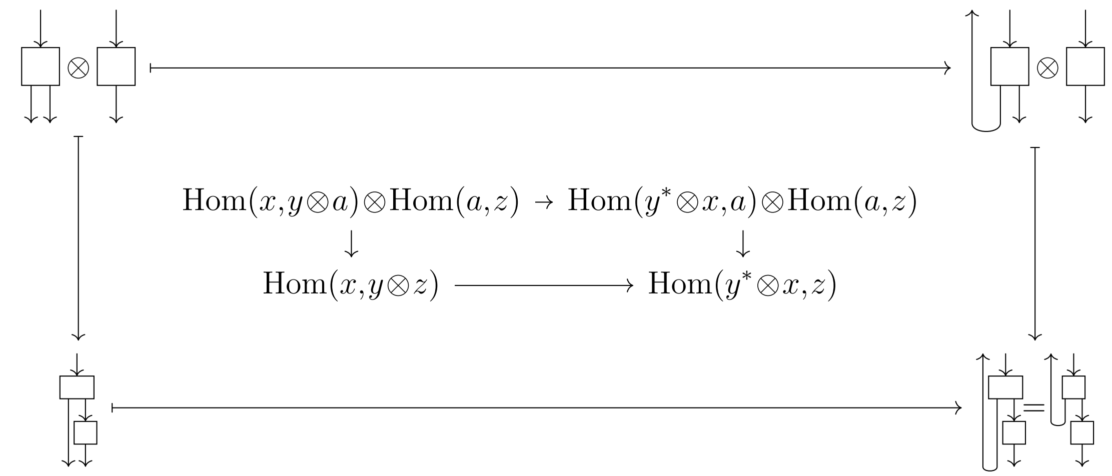
Lemma 2.1. *Let x,y,z be objects in $\mathcal{C}$. For each $a \in I$, the formula $f \otimes g \mapsto (\mathrm{id}_y \otimes g) f$ defines a linear map*

$$\mathrm{Hom}(x, y \otimes a) \otimes \mathrm{Hom}(a, z) \to \mathrm{Hom}(x, y \otimes z).$$

The direct sum of these morphism is an isomorphism

$$\bigoplus_{a \in I} \mathrm{Hom}(x, y \otimes a) \otimes \mathrm{Hom}(a, z) \to \mathrm{Hom}(x, y \otimes z).$$

Proof. For each simple object a, we get a diagram:



The horizontal arrows are the canonical isomorphisms we get from the evaluation. The map on the right-hand side is the composition. We have already seen in Lemma 1.5 that the direct sum over I of these morphisms is an isomorphism. Then, by the commutativity of the diagram, we also know that the direct sum of the morphisms on the left-hand side is an isomorphism. □

Lemma 2.2. *Let x,y,z be objects in $\mathcal{C}$. For each $a \in I$, the formula $f \otimes g \mapsto (g \otimes \mathrm{id}_z) f$ defines a linear map*

$$\mathrm{Hom}(x, a \otimes z) \otimes \mathrm{Hom}(a, y) \to \mathrm{Hom}(x, y \otimes z).$$

The direct sum of these morphisms is an isomorphism

$$\bigoplus_{a \in I} \mathrm{Hom}(x, a \otimes z) \otimes \mathrm{Hom}(a, y) \to \mathrm{Hom}(x, y \otimes z).$$

The proof is analogous to the one of Lemma 2.1.

Definition 2.3. For any $a,b,c \in I$, consider the $\Bbbk$-vector spaces

$$H_c^{ab} = \mathrm{Hom}(c, a\otimes b) \text{ and } H_{ab}^c = \mathrm{Hom}(a\otimes b, c).$$

By Lemma 1.15 the bilinear pairing $\Theta : H_c^{ab} \times H_{ab}^c \to \Bbbk$ is non-degenerate, so we can identify the dual spaces $(H_c^{ab})^*$ and $(H_{ab}^c)^*$ with H_{ab}^c and H_c^{ab} respectively.

Definition 2.4. Let (a,b,c,d,e,f) be an ordered 6-tuple of simple objects. By Lemma 2.1, we get an isomorphism

$$\bigoplus_{f\in I} H_e^{af} \otimes H_f^{bd} \to \mathrm{Hom}(e, a\otimes(b\otimes d))$$

and, by Lemma 2.2, we get an isomorphism

$$\bigoplus_{c\in I} H_e^{cd} \otimes H_c^{ab} \to \mathrm{Hom}(e, (a\otimes b)\otimes d).$$

This yields an isomorphism

$$\bigoplus_{c\in I} H_e^{cd} \otimes H_c^{ab} \to \mathrm{Hom}(e, (a\otimes b)\otimes d) \to \mathrm{Hom}(e, a\otimes(b\otimes d)) \to \bigoplus_{f\in I} H_e^{af} \otimes H_f^{bd},$$

where the second arrow is post-composition with the associator $a_{a,b,d}$. Restricting this morphism to the summand on the left-hand side corresponding to a given $c\in I$ and projecting into the summand on the right-hand side corresponding to a given $f\in I$ yields the *6j-symbol*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+ : H_e^{cd} \otimes H_c^{ab} \to H_e^{af} \otimes H_f^{bd}.$$

Similarly by restricting and projecting the inverse of the above isomorphism, one gets the *opposite 6j-symbol*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_- : H_e^{af} \otimes H_f^{bd} \to H_e^{cd} \otimes H_c^{ab}.$$

Corollary 2.5. *The 6j-symbols correspond to functionals*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+^{\prime} : H_e^{cd} \otimes H_c^{ab} \otimes H_{bd}^f \otimes H_{af}^e \to \Bbbk$$

via the pairing Θ from Lemma 1.15. Similarly, we can define

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_-^{\prime} : H_e^{af} \otimes H_f^{bd} \otimes H_{ab}^c \otimes H_{cd}^e \to \Bbbk$$

for the opposite 6j-symbol. The relationship between these two forms is as follows: Let w and x be elements of H_e^{cd} and H_c^{ab} respectively. We can write

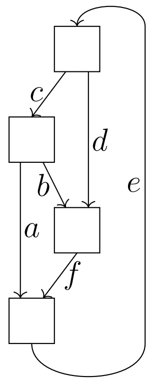
$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+ (w\otimes x) = u\otimes v \in H_e^{af} \otimes H_f^{bd}.$$

Now let y and z be elements of H_{bd}^f and H_{af}^e. Then, we obtain

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+^{\prime} (w\otimes x\otimes y\otimes z) = \mathrm{tr}(zu)\mathrm{tr}(yv).$$

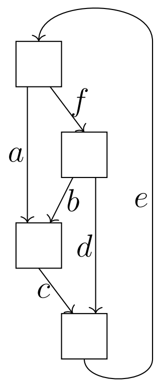
Lemma 2.6. *The linear functionals from Corollary 2.5 are given by*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}'_+ = \dim_q(f)$$



and

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}'_- = \dim_q(c)$$



where on the left and right both sides of the equality are morphisms $H_e^{cd} \otimes H_c^{ab} \otimes H_{bd}^f \otimes H_{af}^e \to \mathbb{k}$ *and respectively* $H_e^{af} \otimes H_f^{bd} \otimes H_{ab}^c \otimes H_{cd}^e \to \mathbb{k}$.

Proof. Let $w \in H_e^{cd}$, $x \in H_c^{ab}$, $y \in H_{bd}^f$ and $z \in H_{af}^e$. The morphism $a_{a,b,d}(x \otimes \mathrm{id}_d)w$ is an element in $\mathrm{Hom}(e, a \otimes (b \otimes d))$. By Lemma 2.1, we can write it as $\sum_{g \in I} (\mathrm{id}_a \otimes v_g) u_g =: \phi$, where $u_g \in H_e^{ag}, v_g \in H_g^{bd}$. It follows from the definition of the 6j-symbol that

$$\begin{Bmatrix} a & b & c \\ d & e & g \end{Bmatrix}_+ (w \otimes x) = u_g \otimes v_g.$$

If we post-compose the ϕ with $\mathrm{id}_a \otimes y$, we get

$$(\mathrm{id}_a \otimes y) a_{a,b,d}(x \otimes \mathrm{id}_d) w = \sum_{g \in I} (\mathrm{id}_a \otimes y v_g) u_g = (\mathrm{id}_a \otimes y v_f) u_f = \dim_q(f)^{-1} \mathrm{tr}(y v_f) u_f.$$

Finally if we rearrange, post-compose with z and take the trace of both sides, we get the desired result

$$\mathrm{tr}(y v_f) \mathrm{tr}(z u_f) = \dim_q(f) \mathrm{tr}(z (\mathrm{id}_a \otimes y) a_{a,b,d}(x \otimes \mathrm{id}_d) w).$$

The proof for the opposite 6j-symbol is analogous. □

Theorem 2.7 (The orthogonality relation)**.** *For all simple objects a, b, c, d, e in $\mathcal{C}$, we have*

$$\sum_{f \in I} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_- \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+ = \mathrm{id}_e^{cd} \otimes \mathrm{id}_c^{ab},$$

where id_c^{ab} *is the identity on* H_c^{ab}.

The proof follows directly from the definition of the 6j-symbols.

Theorem 2.8 (The Biedenharn-Elliot relation)**.** *For all simple objects $a,b,\dots,h,i$ in $\mathcal{C}$, we have*

$$\sum_{j\in I}\left(\mathrm{id}_a^{bh}\otimes\begin{Bmatrix} c & d & j\\ e & h & i\end{Bmatrix}_+\right)\left(\begin{Bmatrix} b & j & g\\ e & a & h\end{Bmatrix}_+\otimes\mathrm{id}_j^{cd}\right)\left(\mathrm{id}_a^{ge}\otimes\begin{Bmatrix} b & c & f\\ d & g & j\end{Bmatrix}_+\right)$$
$$=\left(\begin{Bmatrix} b & c & f\\ i & a & h\end{Bmatrix}_+\otimes\mathrm{id}_i^{de}\right)\left(\mathrm{id}_a^{fi}\otimes P\right)\left(\begin{Bmatrix} f & d & g\\ e & a & i\end{Bmatrix}_+\otimes\mathrm{id}_f^{bc}\right),$$

where P is the standard twist of the two factors of the tensor product.

Proof. We shall prove this theorem via the diagram

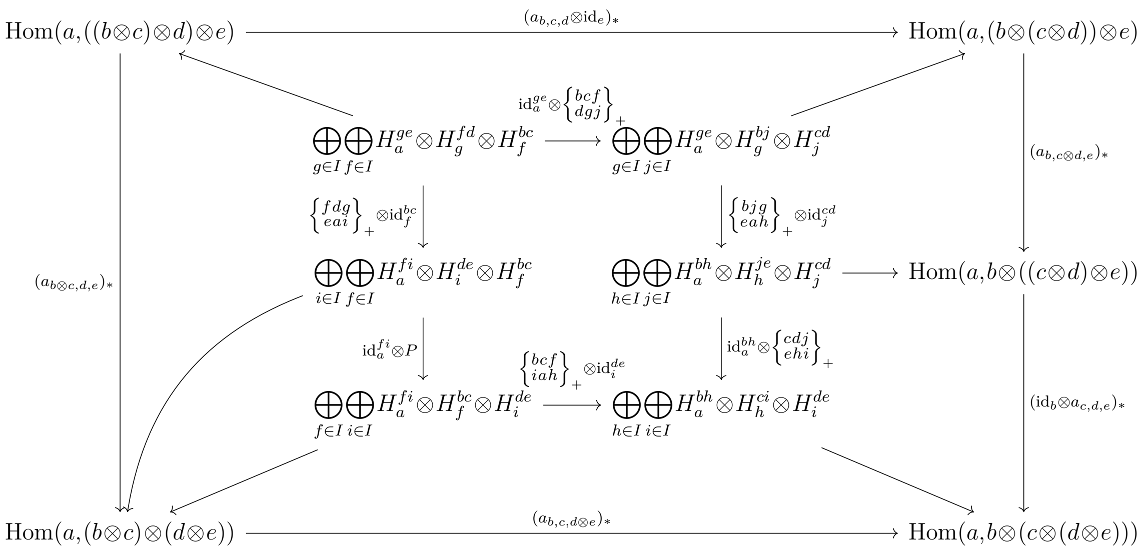
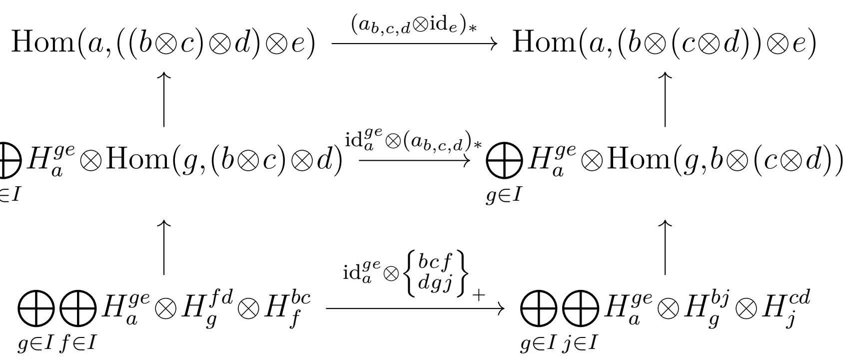


Diagram $*$

where the morphisms from the inner hexagon to the outer pentagon are appropriate combinations of the isomorphisms from Lemmas 2.1 and 2.2. The outer pentagram commutes via the pentagon axiom of the monoidal structure of $\mathcal{C}$. The triangle on the bottom left commutes as permuting the morphisms doesn't change the result of the composition. We consider the top square and prove that it commutes as well:



The bottom square commutes by definition of the 6j-symbol. We can easily verify that the top square of this diagram commutes. Let ϕ and ψ be elements of H_a^{ge} and $\mathrm{Hom}(g,(b\otimes c)\otimes d)$ respectively. We then have:

$$\begin{array}{ccccc}
(\psi\otimes \mathrm{id}_e)\phi & \longmapsto & (a_{b,c,d}\otimes \mathrm{id}_e)(\psi\otimes \mathrm{id}_e)\phi & = & (a_{b,c,d}\psi\otimes \mathrm{id}_e)\phi \\
\uparrow & & & & \uparrow \\
\phi\otimes\psi & & \longmapsto & & \phi\otimes a_{b,c,d}\psi
\end{array}$$

Therefore the top square of diagram $*$ commutes. The proofs for all other squares connecting the inner hexagon to the outer pentagon are analogous. Thus, the inner hexagon commutes because all the morphisms connecting the outer and the inner parts are isomorphisms. By restricting the top left corner of the inner hexagon in diagram $*$ to the summand corresponding to given g and f and projecting the bottom right corner into the summand corresponding to given h and i, we get the result. □

2.1 6j-symbols in $\mathbf{Vect}_G^{\omega,fd}$

To compute the 6j-symbols for $\mathcal{C}=\mathrm{Vect}_G^{\omega,fd}$, note first that $\mathbb{C}_g\otimes\mathbb{C}_h\cong\mathbb{C}_{gh}$, which implies that, for g,h,k in G, $H_k^{g,h}=\mathrm{Hom}(k,g\otimes h)$ is isomorphic to $\mathbb{C}$ if $k=gh$ and $H_k^{g,h}=0$ otherwise. It follows that the 6j-symbols

$$\begin{Bmatrix} g & h & l \\ k & m & n \end{Bmatrix}_{\pm}$$

vanish unless $l=gh$, $m=ghk$ and $n=hk$.

Lemma 2.9. *Let $g,h,k\in G$. The 6j-symbols for $\mathrm{Vect}_G^{\omega,fd}$ take the form*

$$\begin{Bmatrix} g & h & gh \\ k & ghk & hk \end{Bmatrix}_{\pm}=\omega(g,h,k)^{\pm 1}.$$

Proof. As both $H_{ghk}^{gh,k}\otimes H_{gh}^{g,h}$ and $H_{ghk}^{g,hk}\otimes H_{hk}^{h,k}$ are isomorphic to $\mathbb{C}$, the 6j-symbols are given by complex numbers. We now calculate:

$$\begin{array}{ccc}
1 & \in & \mathbb{C} \\
\downarrow & & \downarrow \\
(1\mapsto 1\otimes 1)\otimes(1\mapsto 1\otimes 1) & \in & H_{ghk}^{gh,k}\otimes H_{gh}^{g,h} \\
\downarrow & & \downarrow \\
(1\mapsto(1\otimes 1)\otimes 1) & \in & \mathrm{Hom}(ghk,(g\otimes h)\otimes k) \\
\downarrow & & \downarrow \\
\omega(g,h,k)(1\mapsto 1\otimes(1\otimes 1)) & \in & \mathrm{Hom}(ghk,g\otimes(h\otimes k)) \\
\downarrow & & \downarrow \\
\omega(g,h,k)(1\mapsto 1\otimes 1)\otimes(1\mapsto 1\otimes 1) & \in & H_{ghk}^{g,hk}\otimes H_{hk}^{h,k} \\
\downarrow & & \downarrow \\
\omega(g,h,k) & \in & \mathbb{C}
\end{array}$$

The calculation for the opposite 6j-symbol is analogous. □

3 Turaev-Viro-Barrett-Westbury Invariants

The construction of the invariant in this chapter and the proof are taken from [BW96]. Before we can define the invariant we first need to introduce the topological framework.

A *combinatorial simplicial complex* is a finite set V, whose elements are called vertices, together with a subset $\mathcal{S}$ of the power set $\mathcal{P}(V)$, such that

$$s \in \mathcal{S} \Rightarrow (\forall f \subseteq s : f \in \mathcal{S}).$$

The elements of $\mathcal{S}$ are called *simplices*. A simplex containing $n+1$ elements is called an *n-simplex*. The $(n-1)$-simplices of an n-simplex are called the *faces* of that simplex. We call a complex *n-dimensional* if it contains n-simplices but no $(n+1)$ simplices.

A *simplicial complex* is a combinatorial simplicial complex together with a total ordering on the vertices of each simplex such that the ordering on the vertices on any face of a simplex is the ordering induced from the ordering on the vertices of the simplex. The *i-th face* of an n-simplex s $(0 \le i \le n)$ is the $(n-1)$-simplex s_i obtained by omitting the i-th vertex of s. One could also just require an ordering on the set of all vertices. This would make parts of the proof in this chapter slightly more straightforward but in turn would restrict us when working with manifolds later on.

An *orientation* of an n-dimensional simplicial complex is an assignment of a sign $\varepsilon(s) \in \{-1,+1\}$ to each n-simplex s with the following property: If an $(n-1)$-simplex f is both the i-th face of s and the j-th face of s', then $\varepsilon(s_i) = -\varepsilon(s'_j)$. For each oriented simplicial complex we have 2 possible orientations. If we are given an orientation, we obtain the opposite orientation by assigning the opposite sign to every n-simplex. Note that we require an ordering to define orientations. Without one we could not talk about the i-th face of a simplex.

A *triangulation* of a closed compact orientable 3-manifold X is an oriented 3-dimensional simplicial complex M such that the geometric realisation of M is homeomorphic to X. We will treat simplicial complexes and their geometric realisation interchangeably and therefore we will often call the 1-,2- and 3-simplices of M the *edges*, *faces* and *tetrahedra* of M respectively. The *standard n-simplex* $[0 \ldots n] \subseteq \mathbb{R}^n$ is the convex hull of 0 and the n standard basis-vectors.

Let M be a triangulation of a closed compact orientable 3-manifold and let E be the set of edges of M, F the set of faces, T the set of oriented tetrahedra and v the number of vertices. We call a map $l : E \to I$ a labelling of M. Let $[012]$ be the standard 2-simplex with edges labelled by objects e_{01}, e_{02}, e_{12} in I. To this simplex, we assign the state space $H(012) := \mathrm{Hom}(e_{02}, e_{01} \otimes e_{12}) = H_{e_{02}}^{e_{01}, e_{12}}$. Let $[0123]$ and $[-0123]$ be the standard 3-simplices with either positive or negative orientation with edges labelled $e_{01}, e_{02}, e_{03}, e_{12}, e_{13}, e_{23} \in I$. We assign the *normalised 6j-symbol*

$$J(0123) := \frac{1}{\dim_q(e_{13})} \begin{Bmatrix} e_{01} & e_{12} & e_{02} \\ e_{23} & e_{03} & e_{13} \end{Bmatrix}_+ : H(023) \otimes H(012) \to H(013) \otimes H(123)$$

to the simplex [0123] and *normalised opposite 6j-symbol*

$$J(-0123) := \frac{1}{\dim_q(e_{02})} \begin{Bmatrix} e_{01} & e_{12} & e_{02} \\ e_{23} & e_{03} & e_{13} \end{Bmatrix}_- : H(013)\otimes H(123) \to H(023)\otimes H(012)$$

to the simplex [−0123]. We note that the domain and codomain of $J(0123)$ and $J(-0123)$ are the faces of [0123] and [−0123] with negative and positive orientation respectively. As M is without boundary and oriented, every face of M appears in exactly two adjacent tetrahedra, with opposite induced orientations. This allows us to define the following:

Definition 3.1. Let M be a triangulation of a closed compact orientable 3-manifold with a fixed labelling l. We have the endomorphism

$$\bigotimes_{t\in T} J(t) : \bigotimes_{f\in F} H(f) \to \bigotimes_{f\in F} H(f)$$

up to permutation of the state spaces in the tensor products. The *simplicial invariant of the labelled triangulation M* is defined by

$$Z(M,l) := \mathrm{tr}\left(\bigotimes_{t\in T} J(t)\right).$$

Theorem 3.2. *Let M be a triangulation of a closed compact orientable 3-manifold. Then*

$$C(M) := \dim(\mathcal{C})^{-v} \sum_{l:E\to I} Z(M,l) \prod_{e\in E} \dim_q(l(e)),$$

defines an oriented topological invariant of M: it only depends on the spherical fusion category $\mathcal{C}$ and the homeomorphism class and orientation of M.

The remainder of this chapter is dedicated to the proof of Theorem 3.2. We will show that $C(M)$ is only dependent on the homeomorphism class and orientation of M by showing that it is independent of all arbitrary choices we made when defining $C(M)$, these being the set I of representatives of simple objects and the triangulation of M, the latter of which includes both the choice of a combinatorial simplicial complex and the choice of an ordering on the simplices.

3.1 Invariance under the choice of simple objects

We will show that $C(M)$ does not depend on the choice of simple objects in I. Let I' be another set of representatives of simple objects with a map $i: I \to I'$ such that $i(k) \cong k$ for all $k \in I$. Let $a,\dots,f \in I$ and $\widetilde{a},\dots,\widetilde{f} \in I'$ be simple objects with isomorphisms $\phi_k : k \to \widetilde{k}$ for $k = a,\dots f$.

First we define the morphism:

$$\begin{aligned} \Phi_c^{ab} : \mathrm{Hom}(c, a\otimes b) &\to \mathrm{Hom}(\widetilde{c}, \widetilde{a}\otimes\widetilde{b}) \\ \epsilon &\mapsto (\phi_a\otimes\phi_b)\epsilon\phi_c^{-1} \end{aligned}$$

We will now prove the equation

$$\begin{Bmatrix} \widetilde{a} & \widetilde{b} & \widetilde{c} \\ \widetilde{d} & \widetilde{e} & \widetilde{f} \end{Bmatrix}_+ (\Phi_e^{cd} \otimes \Phi_c^{ab}) = (\Phi_e^{af} \otimes \Phi_f^{bd}) \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_+ \tag{2}$$

with the following diagram.

$$\begin{array}{ccc}
\bigoplus_{c\in I} \mathrm{Hom}(e,c\otimes d)\otimes \mathrm{Hom}(c,a\otimes b) & \xrightarrow{(\Phi_e^{cd}\otimes\Phi_c^{ab})} & \bigoplus_{\tilde{c}\in I'} \mathrm{Hom}(\widetilde{e},\widetilde{c}\otimes\widetilde{d})\otimes\mathrm{Hom}(\widetilde{c},\widetilde{a}\otimes\widetilde{b}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(e,(a\otimes b)\otimes d) & \xrightarrow{\epsilon\mapsto((\phi_a\otimes\phi_b)\otimes\phi_d)\epsilon\phi_e^{-1}} & \mathrm{Hom}(\widetilde{e},(\widetilde{a}\otimes\widetilde{b})\otimes\widetilde{d}) \\
\downarrow (a_{a,b,d})_* & & \downarrow (a_{\tilde{a},\tilde{b},\tilde{d}})_* \\
\mathrm{Hom}(e,a\otimes(b\otimes d)) & \xrightarrow{\epsilon\mapsto(\phi_a\otimes(\phi_b\otimes\phi_d))\epsilon\phi_e^{-1}} & \mathrm{Hom}(\widetilde{e},\widetilde{a}\otimes(\widetilde{b}\otimes\widetilde{d})) \\
\uparrow & & \uparrow \\
\bigoplus_{f\in I} \mathrm{Hom}(e,a\otimes f)\otimes\mathrm{Hom}(f,b\otimes d) & \xrightarrow{(\Phi_e^{af}\otimes\Phi_f^{bd})} & \bigoplus_{\tilde{f}\in I'} \mathrm{Hom}(\widetilde{e},\widetilde{a}\otimes\widetilde{f})\otimes\mathrm{Hom}(\widetilde{f},\widetilde{b}\otimes\widetilde{d})
\end{array}$$

The unlabelled morphisms are the isomorphisms from the lemmas 2.1 and 2.2. We will show that the top square commutes. The middle square commutes by the naturality of the associator, the bottom one analogously to the top square. Let $\epsilon\otimes\eta$ be an element of $\mathrm{Hom}(e,c\otimes d)\otimes\mathrm{Hom}(c,a\otimes b)$, then the top square is given by:

$$\begin{array}{ccc}
\epsilon\otimes\eta & \longmapsto & (\phi_c\otimes\phi_d)\epsilon\phi_e^{-1}\otimes(\phi_a\otimes\phi_b)\eta\phi_c^{-1} \\
\big\downarrow & & \downarrow \\
 & & ((\phi_a\otimes\phi_b)\eta\phi_c^{-1}\otimes\mathrm{id}_{\tilde{d}})(\phi_c\otimes\phi_d)\epsilon\phi_e^{-1} \\
 & & \| \\
 & & ((\phi_a\otimes\phi_b)\eta\otimes\phi_d)\epsilon\phi_e^{-1} \\
 & & \| \\
(\eta\otimes\mathrm{id}_d)\epsilon & \longmapsto & ((\phi_a\otimes\phi_b)\otimes\phi_d)(\eta\otimes\mathrm{id}_d)\epsilon\phi_e^{-1}
\end{array}$$

Therefore the diagram commutes which implies equation 2.

Let t be a 3-simplex in M, $l: E\to I$ a fixed labelling. We define $J'(t)$ as the normalised 6j-symbol for the labelling $l' = i\circ l$. Let f be a 2-simplex in M. We define $\Phi(f)$ as Φ_c^{ab} for $H(f) = H_c^{ab}$. Thanks to equation 2, we have:

$$\bigotimes_{t\in T} J'(t) = \bigotimes_{f\in F}\Phi(f)\bigotimes_{t\in T}J(t)\bigotimes_{f\in F}\Phi(f)^{-1}$$

This implies

$$\begin{aligned}
Z(M,l') &= \mathrm{tr}\left(\bigotimes_{t\in T}J'(t)\right) = \mathrm{tr}\left(\bigotimes_{f\in F}\Phi(f)\bigotimes_{t\in T}J(t)\bigotimes_{f\in F}\Phi(f)^{-1}\right) \\
&= \mathrm{tr}\left(\bigotimes_{t\in T}J(t)\bigotimes_{f\in F}\Phi(f)^{-1}\bigotimes_{f\in F}\Phi(f)\right) = \mathrm{tr}\left(\bigotimes_{t\in T}J(t)\right) = Z(M,l)
\end{aligned}$$

This proves that $C(M)$ does not depend on the choice of simple objects.

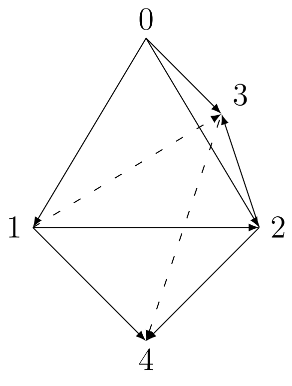
3.2 Invariance under the choice of ordering

In this chapter we are going to show that $C(M)$ does not depend on the choice of orderings on the simplices of the simplicial complex. Every change of orderings induces a change of the orderings of the simplices of any subcomplex. Firstly, we will show how these changes affect the orientation of 3-simplices.

Lemma 3.3. *A change of orderings preserves the condition that a face shared by two adjacent tetrahedra has opposite induced orientations, by reversing the orientation of a 3-simplex if it acts on its ordering as a permutation with odd parity and preserving its orientation otherwise.*

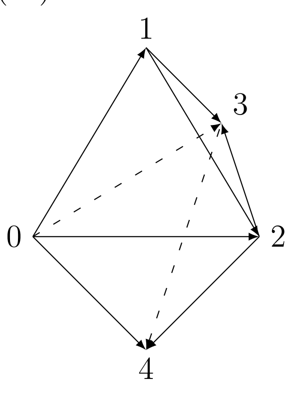
Proof. It is sufficient to show how the these changes act on a 3-simplex and another adjacent 3-simplex, i.e. a triagonal bipyramid, as we only need to check that the orientations of a 2-simplex induced by its two adjacent 3-simplices are opposite of each other after the change. We note that the possible orderings on a tetrahedron correspond to elements of S_4 and orderings on a triagonal bipyramid correspond to elements of S_5. The changes of orderings on the bipyramid also correspond to elements of S_5 acting on the given orderings. The group S_5 itself now acts transitively on the set of orderings. Therefore, it's sufficient to let the permutation act on only one choice of initial orderings. Furthermore, as S_5 is generated by (01) and (01234), it's sufficient to analyse how these generators act on the orientations of the 3-simplices.

Initial ordering:



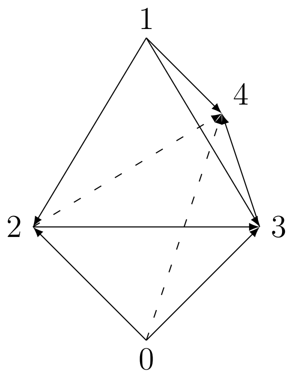
The tetrahedra have either both positive or both negative orientations. The shared 2-face has opposite orientations in the two tetrahedra.

(01):



(01) acts on the top tetrahedron as the odd permutation (01) and changes its orientation, and acts as the identity on the bottom tetrahedron. The shared 2-face has opposite orientations due to the change of the top tetrahedron's orientation.

(01234):



(01234) acts on the top tetrahedron as the identity and on the bottom tetrahedron like the odd permutation (0123) and changes its orientation. The shared 2-face has opposite orientations due to the change of the bottom tetrahedron's orientation.

□

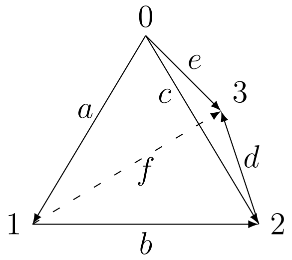
Lemma 3.4. *Let t be 3-simplex with an ordering and labels $a,\ldots,f \in I$ associated to the edges of t. For each permutation acting on the ordering of t, there is a canonical re-labelling of t such that the normalised 6j-symbols associated to the different simplicial complexes are canonically isomorphic.*

This lemma proves that $C(M)$ is invariant under a change of orderings as the canonical re-labelling preserves the dimensions of simple objects and thus only permutes the summands of the sum in the definition of $C(M)$.

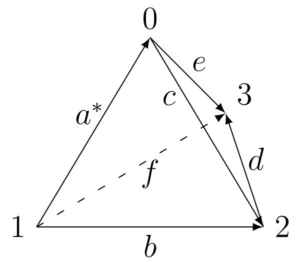
Proof. The normalised 6j-symbols correspond to normalised functional 6j-symbols analogously to those from Corollary 2.5. We will use Lemma 2.6 to show that these normalised functional 6j-symbols are canonically isomorphic, which implies that the usual normalised 6j symbols are canonically isomorphic as well.

Similarly to the previous proof, the possible orderings on a 3-simplex correspond to elements of S_4. The group S_4 acts transitively on the set of orderings on a 3-simplex. It is therefore sufficient to let the permutation act on only one initial ordering. As S_4 is generated by the permutations (01) and (0123), it is sufficient to determine how these generators act on the associated 6j-symbols. We will re-label any edge whose direction has been flipped by assigning it the dual of its original label.

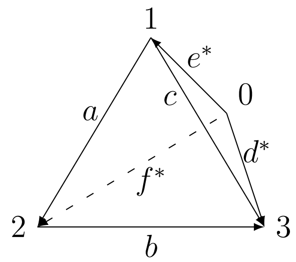
Initial ordering



(01):



(0123):



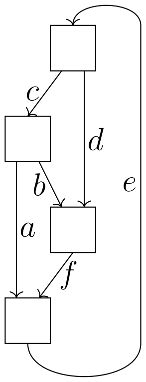
To these, we assign the following normalised functional 6j-symbols:

$$\text{Initial ordering: } \frac{1}{\dim_q(f)} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}'_+ : H_e^{cd} \otimes H_c^{ab} \otimes H_{bd}^f \otimes H_{af}^e \to \mathbb{k}$$

$$(01): \frac{1}{\dim_q(b)} \begin{Bmatrix} a^* & c & b \\ d & f & e \end{Bmatrix}'_- : H_f^{a^*e} \otimes H_e^{cd} \otimes H_{a^*c}^b \otimes H_{bd}^f \to \mathbb{k}$$

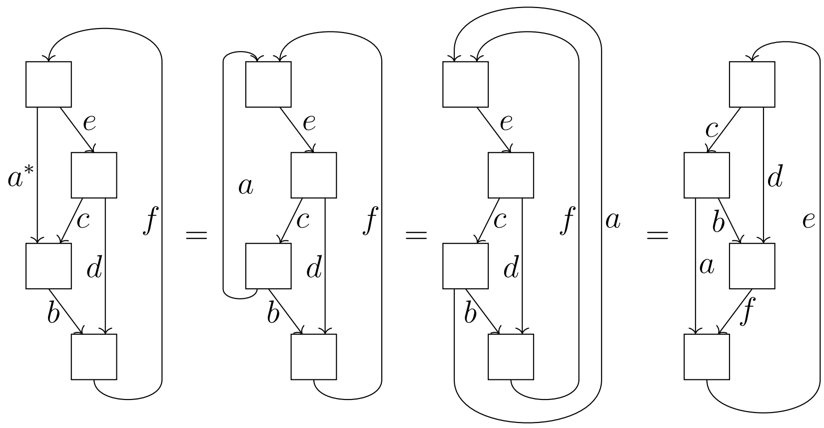
$$(0123): \frac{1}{\dim_q(f^*)} \begin{Bmatrix} e^* & a & f^* \\ b & d^* & c \end{Bmatrix}'_- : H_{d^*}^{e^*c} \otimes H_c^{ab} \otimes H_{e^*a}^{f^*} \otimes H_{f^*b}^{d^*} \to \mathbb{k}$$

Lemma 2.6 tells us that the first functional corresponds to the functional



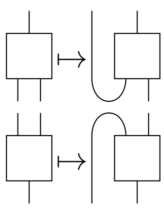
$$: H_e^{cd} \otimes H_c^{ab} \otimes H_{bd}^f \otimes H_{af}^e \to \mathbb{k}.$$

We will show that the other functionals are also canonically isomorphic to this. The second functional is equivalent to



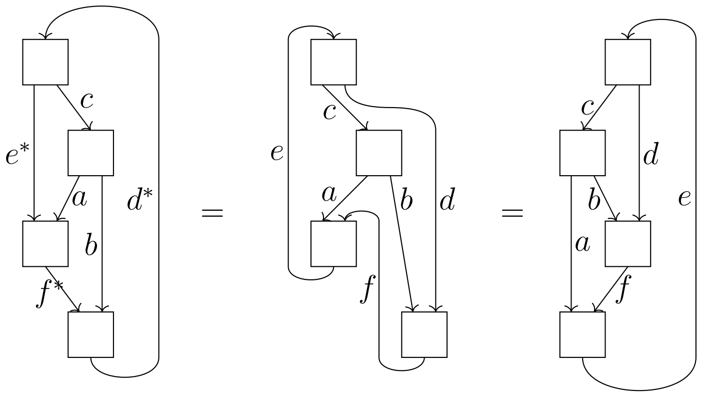
In the first equality, we use the canonical isomorphisms

$$H_f^{a^*e} \to H_{af}^e, \qquad H_{a^*c}^b \to H_c^{ab},$$



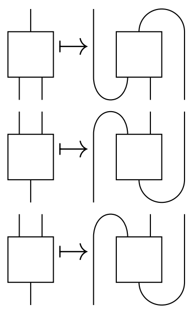
We took use of $\mathrm{tr}_L = \mathrm{tr}_R$ in the second equality and the invariance of the trace under cyclic permutation in the last equality.

The third functional is equivalent to



The first step follows from the canonical isomorphisms

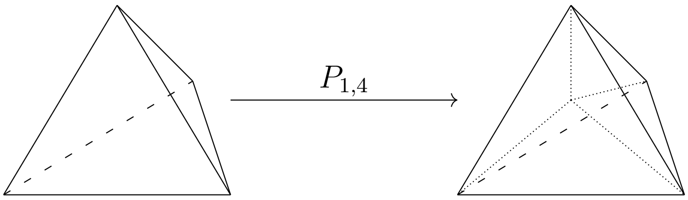
$$H_{d^*}^{e^*c} \to H_e^{cd}, \qquad H_{e^*a}^{f^*} \to H_{af}^{e}, \qquad H_{f^*b}^{d^*} \to H_{bd}^{f},$$



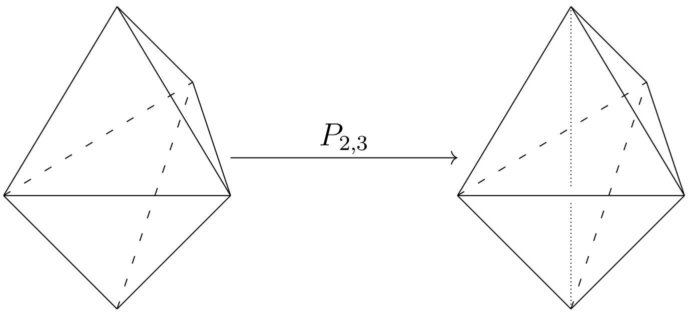
In the second equality we shift around some of the boxes and use $\mathrm{tr}_L = \mathrm{tr}_R$. $\square$

3.3 Invariance under the choice of triangulation

Thanks to the work of Pachner [Pac91], we know that two triangulations are homeomorphic if they can be related by a finite number of Pachner moves. In the 3-dimensional case, we have the 1-4 Pachner move



that replaces a tetrahedron by four tetrahedra sharing a common vertex, and the 2-3 Pachner move



which replaces two tetrahedra with a common face by three tetrahedra with a common edge. We also have the 4-1 and the 3-2 Pachner moves which are the inverse to the 1-4 and 2-3 moves respectively. Hence, if $C(M)$ is invariant under $P_{1,4}$ and $P_{2,3}$, then it only depends on the homeomorphism class and orientation of M and is a topological invariant.

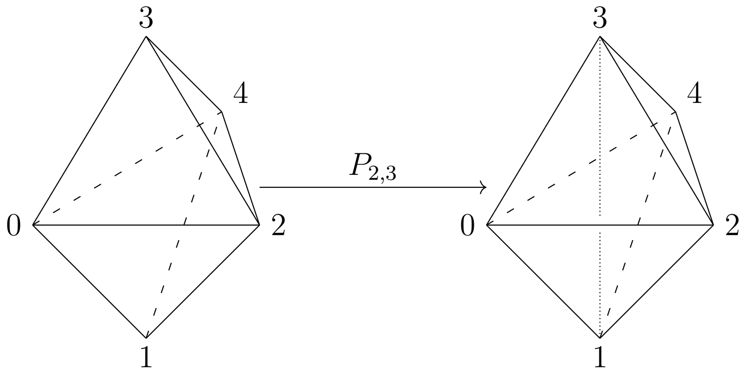
The Pachner moves also allow us to expand framework to allow semisimplicial complexes. Every semisimplicial complex is homeomorphic to a simplicial complex via barycentric subdivision. We know that these complexes must be related via Pachner moves, so the invariant associated to the semisimplicial complex must coincide with the invariant for the simplicial complex.

Invariance under $P_{2,3}$

Let M be a triangulation with edge set E and M' the triangulation with edge set $E'=E\cup\{e'\}$ obtained by applying $P_{2,3}$ to M. We now want to show:

$$\begin{aligned} C(M') &= \dim(\mathcal{C})^{-v} \sum_{l:E'\to I} Z(M',l) \prod_{e\in E'} \dim_q(l(e)) \\ &= \dim(\mathcal{C})^{-v} \sum_{l:E\to I} \sum_{i\in I} Z(M',l') \prod_{e\in E} \dim_q(l(e)) \dim_q(i) \\ &\overset{!}{=} \dim(\mathcal{C})^{-v} \sum_{l:E\to I} Z(M,l) \prod_{e\in E} \dim_q(l(e)) = C(M) \end{aligned}$$

Thus, it suffices to show that $Z(M,l)=\sum_{i\in I}\dim_q(i)Z(M',l')$, where l' and l agree on E and $l'(e')=i$. Locally, we can represent the action of $P_{2,3}$ as



Due to the Biedenharn-Elliot relation, we get

$$\begin{aligned}&(J(0124)\otimes \mathrm{id}_{H(234)})(\mathrm{id}_{H(024)}\otimes P)(J(0234)\otimes \mathrm{id}_{H(012)})\\ &=\sum_{e_{13}\in I}\dim_q(e_{13})(\mathrm{id}_{H(014)}\otimes J(1234))(J(0134)\otimes \mathrm{id}_{H(123)})(\mathrm{id}_{H(034)}\otimes J(0123)).\end{aligned}$$

The desired result follows from linear algebra. We start by writing the trace over all 6j-symbols as the trace over the partial trace over the factor $H(024)$. By choosing bases and appropriate dual bases we can rearrange $\mathrm{tr}_{H(024)}(J(0124)\otimes J(0234))$ into the left hand side of the Biedenharn-Elliot relation. The right hand side follows similarly.

Invariance under $P_{1,4}$

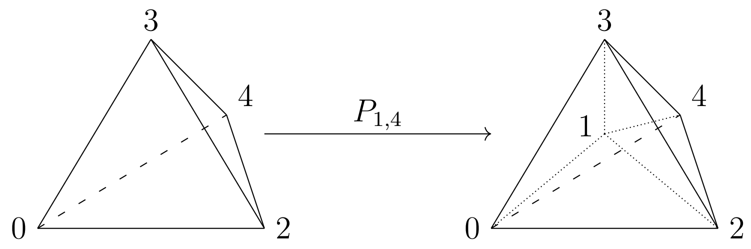
Let M be a triangulation with edge set E and v vertices. Let M' be the triangulation with edge set $E'=E\cup\{e_1,e_2,e_3,e_4\}$ and $v+1$ vertices obtained by applying $P_{1,4}$ to M. We now want to show that

$$\begin{aligned}C(M')&=\dim(\mathcal{C})^{-(v+1)}\sum_{l:E'\to I}Z(M',l)\prod_{e\in E'}\dim_q(l(e))\\ &=\dim(\mathcal{C})^{-v}\dim(\mathcal{C})^{-1}\sum_{l:E\to I}\sum_{\substack{i_1,i_2,\\ i_3,i_4\in I}}Z(M',l')\prod_{e\in E}\dim_q(l(e))\prod_{k=1}^{4}\dim_q(i_k)\\ &\overset{!}{=}\dim(\mathcal{C})^{-v}\sum_{l:E\to I}Z(M,l)\prod_{e\in E}\dim_q(l(e))=C(M).\end{aligned}$$

Hence, it suffices to show that

$$Z(M,l)=\dim(\mathcal{C})^{-1}\sum_{\substack{i_1,i_2,\\ i_3,i_4\in I}}Z(M',l')\prod_{k=1}^{4}\dim_q(i_k),$$

where l and l' agree on E and $l'(e_k)=i_k$ for $k\in\{1,2,3,4\}$. Locally, we can represent the action of $P_{2,3}$ as



where the bottom tetrahedron $[-0124]$ has negative orientation. The equation

$$J(0234) = \dim(\mathcal{C})^{-1} \sum_{\substack{e_{01},e_{12},\\ e_{13},e_{14}\in I}} \Big(\mathrm{tr}_3((1\otimes P)(J(-0124)\otimes 1)(1\otimes J(1234)) (J(0134)\otimes 1)(1\otimes J(0123) \prod_{k=0,2,3,4} \dim_q(e_{1k})\Big), \qquad (3)$$

shows that $C(M)$ is invariant under $P_{1,4}$, where tr_3 is the partial trace over the third factor and 1 is the appropriate identity morphism. This follows from linear algebra similarly to the proof for $P(2,3)$. We will prove equation 3 in the following:

We will start with the Biedenharn-Eilliot relation:

$$(J(0124)\otimes 1)(1\otimes P)(J(0234)\otimes 1) = \sum_{e_{13}\in I} \dim_q(e_{13})(1\otimes J(1234))(J(0134)\otimes 1)(1\otimes J(0123))$$

We now post-compose with $(1\otimes P)(J(-0124)\otimes 1)$, multiply both sides with $\dim(\mathcal{C})^{-1}$ $\dim_q(e_{01})\dim_q(e_{12})\dim_q(e_{14})$, take the trace over the third factor and sum over $e_{01},e_{12},e_{14}\in I$. This yields the equation

$$\begin{aligned}
&\dim(\mathcal{C})^{-1} \sum_{e_{01},e_{12},e_{14}\in I} \Big(\mathrm{tr}_3((1\otimes P)(J(-0124)\otimes 1)(J(0124)\otimes 1)(1\otimes P) (J(0234)\otimes 1))\dim_q(e_{01})\dim_q(e_{12})\dim_q(e_{14})\Big)\\
&= \dim(\mathcal{C})^{-1} \sum_{\substack{e_{01},e_{12},\\ e_{13},e_{14}\in I}} \Big(\mathrm{tr}_3((1\otimes P)(J(-0124)\otimes 1)(1\otimes J(1234)) (J(0134)\otimes 1)(1\otimes J(0123) \prod_{k=0,2,3,4} \dim_q(e_{1k})\Big).
\end{aligned}$$

The right-hand side of this equation is the right-hand side of equation 3. We can rewrite the left-hand side to

$$\dim(\mathcal{C})^{-1} \sum_{e_{01},e_{12}\in I} \Big(\mathrm{tr}_3((1\otimes P)(\sum_{e_{14}\in I} \dim_q(e_{14})(J(-0124)J(0124))\otimes 1) (1\otimes P)(J(0234)\otimes 1))\dim_q(e_{01})\dim_q(e_{12})\Big).$$

Using the orthogonality relation, this becomes

$$
\begin{aligned}
&\frac{\dim(\mathcal{C})^{-1}}{\dim_q(e_{02})}\sum_{e_{01},e_{12}\in I}\dim_q(e_{01})\dim_q(e_{12})\mathrm{tr}_3((J(0234)\otimes 1))\\
=&\frac{\dim(\mathcal{C})^{-1}}{\dim_q(e_{02})}\sum_{e_{01},e_{12}\in I}\dim_q(e_{01})\dim_q(e_{12})\dim\mathrm{Hom}(e_{02},e_{01}\otimes e_{12})J(0234)\\
=&\frac{\dim(\mathcal{C})^{-1}}{\dim_q(e_{02})}\sum_{e_{01},e_{12}\in I}\dim_q(e_{01})\dim_q(e_{12})\dim\mathrm{Hom}(e_{01},e_{02}\otimes e_{12}^*)J(0234)\\
=&\frac{\dim(\mathcal{C})^{-1}}{\dim_q(e_{02})}\sum_{e_{12}\in I}\dim_q(e_{12})\dim_q(e_{02}\otimes e_{12}^*)J(0234)\\
=&\dim(\mathcal{C})^{-1}\sum_{e_{12}\in I}\dim_q(e_{12})^2 J(0234)=\dim(\mathcal{C})^{-1}\dim(\mathcal{C})J(0234)=J(0234)
\end{aligned}
$$

where the second equality follows from $\mathrm{Hom}(e_{02},e_{01}\otimes e_{12})\cong\mathrm{Hom}(e_{01}\otimes e_{12},e_{02})\cong\mathrm{Hom}(e_{01},e_{02}\otimes e_{12}^*)$ and the third from Lemma 1.18. This proves the claim.

4 Computations of Invariants for $\mathcal{C} = \mathbf{Vect}_G^{\omega,fd}$ and the Lens Spaces

In the following, we work with the spherical fusion category $\mathrm{Vect}_G^{\omega,fd}$ of finite dimensional G-graded vector spaces over $\mathbb{C}$ and derive a formula for Turaev-Viro-Barrett-Westbury invariants of lens spaces $L(p,q)$. The ideas on how to triangulate $L(p,q)$ and how to simplify its Turaev-Viro-Barrett-Westbury invariants are taken from [AC93]. With our work in sections 1.1 and 2.1, we can now arrive at the following version of Theorem 3.2.

Corollary 4.1. *As* $\dim_q(g) = 1$ *for all* g *in* G*, the normalised 6j-symbols are given by* $\omega(g,h,k)^{\pm 1}$ *and the Turaev-Viro-Barrett-Westbury invariants are*

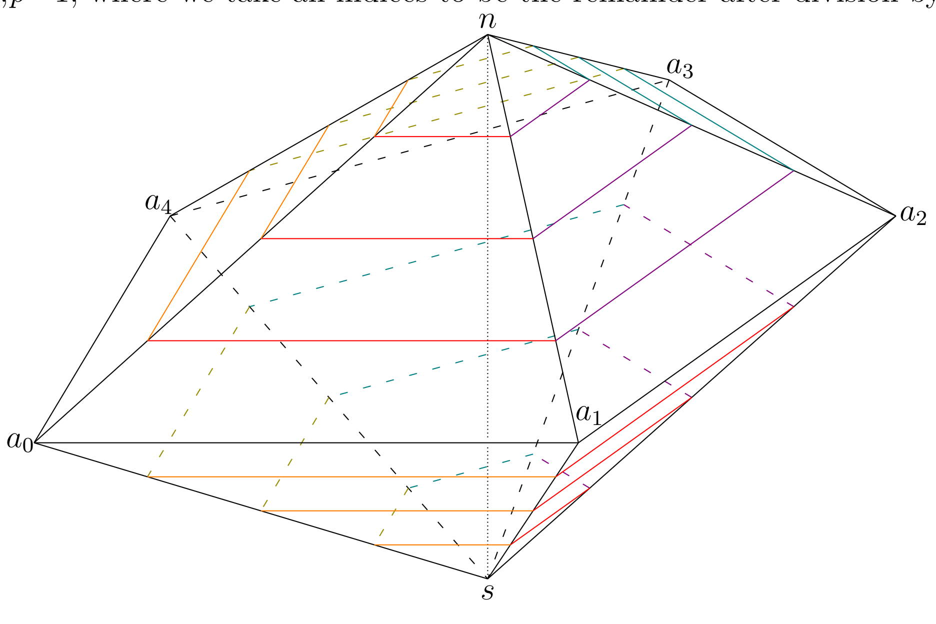
$$C(M) = |G|^{-v} \sum_{\substack{l:E\to G \\ Z(M,l)\neq 0}} \prod_{t\in T} \omega(l(t_{01}), l(t_{12}), l(t_{23}))^{\varepsilon(t)},$$

where t_{ij} *is the edge of the tetrahedron* t *connecting the* i*-th and* j*-th vertex and* $\varepsilon(t)$ *is the orientation of* t*.*

4.1 The lens spaces $L(p,q)$

Definition 4.2. Let $p \in \mathbb{N}$ and $q \in \{1,\dots,p-1\}$ such that p and q are coprime. The lens space $L(p,q)$ is obtained as follows:

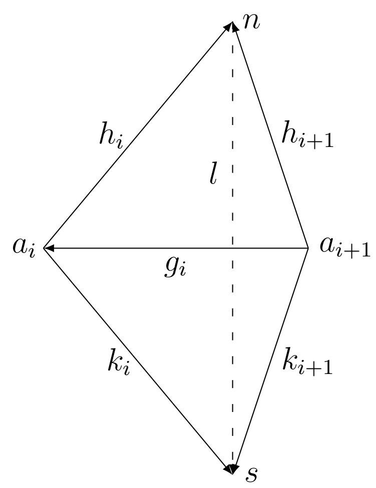
Take a p-sided regular polygon with vertices labelled $a_0,\dots,a_{p-1}$ and put two points n and s directly above and below the center of the polygon respectively. This yields a solid bipyramid. Now identify the triangles $[a_{i+1}, a_i, n]$ with $[a_{q+i+1}, a_{q+i}, s]$ for each $i = 0,\dots,p-1$, where we take all indices to be the remainder after division by p.



The lens space $L(5,1)$, where we identify those sides with the same colours

This yields a triangulation of the lens space $L(p,q)$ with the tetrahedra $[a_{i+1},a_i,n,s]$, where all tetrahedra have positive orientation. We denote the the lens space where every tetrahedron has negative orientation by $-L(p,q)$.

We now assign group elements g_i to the edge $[a_{i+1},a_i]$, h_i to $[a_i,n]$, k_i to $[a_i,s]$ and l to $[n,s]$ for all $i=0,\dots,p-1$. The labelled tetrahedron $[a_{i+1},a_i,n,s]$ now looks like the following:



As the edge $[a_{i+1},a_i]$ is identified with $[a_{q+i+1},a_{q+i}]$, the labels must satisfy $g_i=g_{q+i}$. Since p and q are coprime, it follows that $g_i=g$ for all $i\in\{0,\dots,p-1\}$. For the state space $H^{g,h_i}_{h_{i+1}}$ to be nonzero, we need $gh_i=h_{i+1}$. This implies $h_i=g^ih_0$ and $g^p=1$ and we write $h=h_0$. Analogously, we get $k_i=g^ik_0$ and we write $k=k_0$. Identifying $[a_i,n]$ with $[a_{q+i},s]$ leads to the requirement $h=k_q=g^qk\Leftrightarrow k=g^{-q}h$. For the state space $H^{h_i,l}_{k_i}$ to be nonzero, we need $h_il=k_i$ which implies $l=h^{-1}g^{-q}h$. By fixing $g,h\in G$ with $g^p=e$, we get a labelling l such that

$$Z(L(p,q),l)=\prod_{i=0}^{p-1}\omega(g,g^ih,h^{-1}g^{-q}h),$$

and $Z(L(p,q),l)=0$ for all labellings that are not of this form.

We can now simplify the product above to show that it is independent of h. If we take the cocycle condition from equation 1 for $(g,h,k,l)\equiv(g,g^i,g^{-q},h)$, divide it by the cocycle condition for $(g,h,k,l)\equiv(g,g^i,h,h^{-1}g^{-q}h)$ and take the product over $i=0,\dots,p-1$ on both sides, we get

$$\prod_{i=0}^{p-1}\frac{\omega(g^{i+1},g^{-q},h)\omega(g,g^i,g^{-q}h)}{\omega(g^{i+1},h,h^{-1}g^{-1}h)\omega(g,g^i,g^{-q}h)}=\prod_{i=0}^{p-1}\frac{\omega(g,g^i,g^{-q})\omega(g,g^{i-q},h)\omega(g^i,g^{-q},h)}{\omega(g,g^i,h)\omega(g,g^ih,h^{-1}g^{-q}h)\omega(g^i,h,h^{-1}g^{-q}h)}$$

We see, that

- on the left-hand side, we have $\omega(g,g^i,g^{-q}h)$ both in the numerator and denominator,

- on the right-hand side, we have an $\omega(g,g^i,h)$ in the denominator for each $\omega(g,g^{i-q},h)$ in the numerator,
- for each $\omega(g^{i+1},g^{-q},h)$ in the numerator on the left, we have an $\omega(g^i,g^{-q},h)$ in the numerator on the right and
- for each $\omega(g^{i+1},h,h^{-1}g^{-1}h)$ in the denominator on the right, we have $\omega(g^i,h,h^{-1}g^{-q}h)$ in the denominator on the left.

We can now cancel these terms and rearrange the equation to

$$\prod_{i=0}^{p-1}\omega(g,g^ih,h^{-1}g^{-q}h)=\prod_{i=0}^{p-1}\omega(g,g^i,g^{-q}).$$

We now have

$$\begin{aligned}C(L(p,q)) &= |G|^{-(p+2)}\sum_{\substack{g,h\in G\\ g^p=1}}\prod_{i=0}^{p-1}\omega(g,g^ih,h^{-1}g^{-q}h)\\ &= |G|^{-(p+2)}\sum_{\substack{g,h\in G\\ g^p=1}}\prod_{i=0}^{p-1}\omega(g,g^i,g^{-q}) = |G|^{-(p+1)}\sum_{\substack{g\in G\\ g^p=1}}\prod_{i=0}^{p-1}\omega(g,g^i,g^{-q})\end{aligned}$$

4.2 The lens spaces $L(5,1)$ and $L(5,2)$

We will now show that Turaev-Viro-Barret-Westbury invariants can distinguish non-homeomorphic topological spaces with the same homology and homotopy groups by looking at the spaces $L(5,1)$ and $L(5,2)$. Let $G = \mathbb{Z}/5\mathbb{Z}$. Every element of $\mathbb{Z}/5\mathbb{Z}$ satisfies the condition $5g = 0$. From [EGNO15, equ 2.34], we know that the 3-cocylces on $\mathbb{Z}/5\mathbb{Z}$ are given by

$$\omega(g,h,k) = \exp\left(\frac{2\pi i r[g]([h]+[k]-[h+k])}{25}\right)$$

for $r \in \{0,1,2,3,4\}$, where $[\cdot]$ maps each element of $\mathbb{Z}/5\mathbb{Z}$ onto its smallest non-negative representative in $\mathbb{Z}$. For $r = 1$ we have

$$\begin{aligned}
C(L(5,q)) &= \frac{1}{5^6}\sum_{g=0}^{4}\prod_{j=0}^{4}\exp\left(\frac{2\pi i g([jg]+[-qg]-[(j-q)g])}{25}\right) \\
&= \frac{1}{5^6}\sum_{g=0}^{4}\exp\left(\frac{2\pi i g}{25}\sum_{j=0}^{4}[jg]+[-qg]-[(j-q)g]\right) \\
C(-L(5,q)) &= \frac{1}{5^6}\sum_{g=0}^{4}\exp\left(-\frac{2\pi i g}{25}\sum_{j=0}^{4}[jg]+[-qg]-[(j-q)g]\right),
\end{aligned}$$

so the invariant of $L(5,q)$ with opposite orientation is the complex conjugate of $C(L(5,q))$. If we now put $q = 1,2$ into the above formula, one can compute

$$C(L(5,1)) = C(-L(5,1)) = \frac{\sqrt{5}}{5^6} \neq -\frac{\sqrt{5}}{5^6} = C(L(5,2)) = C(-L(5,2)).$$

If $L(5,1)$ and $L(5,2)$ were homeomorphic, we could choose the same triangulation and orientation for both of them and their invariants would be equal. But since the invariant of $L(5,1)$ with any of the two possible orientations is not equal to the invariant of $L(5,2)$ with any orientation, $L(5,1)$ and $L(5,2)$ are not homeomorphic.

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